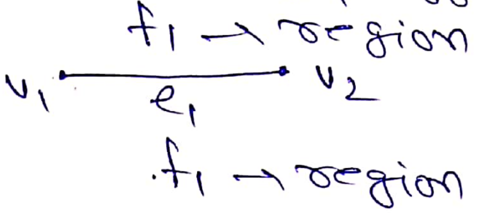
Theorems on Planar Graphs

Thm 1) Euler's Formula for Planar Graph

A connected planar graph with 'n' vertices and 'e' edges has $(e-n+2)$ regions

Pf → Let G be a connected planar graph we shall prove Euler's formula by induction on the number of edges. of G.

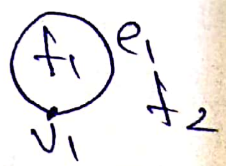
If $e=1$ then $n=1$ or 2 as below—



graph 1.1

In this case, the number of regions $= e-n+2 = 1-2+2=1$ which is the case as in above graph.

If $e=1$ and $n=1$ then the vertex has self loop as—



graph 1.2

Then $e-n+2 = 1-1+2=2$

i.e. 2 regions., which is true as shown in above graph 1.2

∴ the result is true for $e=1$

Now we assume that the result is true for all graphs with at most $(e-1)$ edges. we shall show that the theorem is true for 'e' edges.

Let G be a connected graph with 'e' edges and f regions.
If G is a ~~tree~~ tree then number of edges $e = n-1$ and numbers of regions is 1 as there is no circuit in tree and infinite region will be the only region. Then by Euler's formula, the

number of regions $= e-n+2$

$$= (n-1)-n+2$$
$$= n-1-n+2$$
$$= 1$$

$\therefore$ Theorem is true in this case.

If G is not a tree then it has some circuits. Let 'a' be an edge in some circuit. Removal of 'a' from the plane representation of G will merge the two regions into one region. $\therefore$ (G−a) is a connected graph with 'n' vertices, (e−1) edges and f−1 regions. Here 'f' is the number of regions in G. Then by induction hypothesis, we have—

$$f-1 = e-n+2$$
$$\Rightarrow f-1 = (e-1)-n+2 \quad , \text{as } e=e-1 \text{ in } G-a$$
$$\Rightarrow f = e-n+2$$

Hence the theorem.

Ex.→ If every region of a simple planar graph with 'n' vertices and 'e' edges is bounded by 'k' edges then to show that–

$$e = \frac{k(n-2)}{(k-2)}$$

Soln.→ Let 'f' be the no. of regions in the graph.

Since every region is bounded by 'k' edges, ∴ we have–

$$kf = 2e \quad \rightarrow (1)$$

From Euler's formula →

$$f = e - n + 2 \quad \rightarrow (2)$$

Put value of 'f' from (1) in (2), we get

$$\frac{2e}{k} = e - n + 2$$

$$2e = ke - nk + 2k$$

$$2e - ke = 2k - nk$$

$$(2-k)e = (2-n)k$$

or $$e = \frac{(2-n)}{(2-k)}k$$

or $$\boxed{e = \frac{k(n-2)}{(k-2)}}$$

(Proved)

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Ex: In a simple planar graph G of n vertices ($n \geq 3$), there is at least one vertex of degree ≤ 5.

Soln: Let G be a simple planar graph with 'n' vertices and 'e' edges.

If possible, we assume that degree of each vertex of G is ≥ 5.

Then we have →

$$2e \geq 6n$$ as sum of degree of all vertices of G is at least 6n.

or $e \geq 3n$ → (1)

But from Euler's formula, we know that →

$$e \leq 3n - 6 \rightarrow (2)$$

Eqn's (1) & (2) contradict each other. ∴ at least one vertex of G must have degree ≤ 5.

(Proved)